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## THE DESCRIPTIVE LOOK AT THE SIZE OF SUBSETS OF GROUPS

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ABSTRACT. We explore the Borel complexity of some basic families of subsets of a countable group (large, small, thin, sparse and other) defined by the size of their elements. Applying the obtained results to the Stone-Čech compactification  $\beta G$  of  $G$ , we prove, in particular, that the closure of the minimal ideal of  $\beta G$  is of type  $F_{\sigma\delta}$ .

Given a group  $G$ , we denote by  $\mathbf{P}_G$  and  $\mathbf{F}_G$  the Boolean algebra of all subsets of  $G$  and its ideal of all finite subsets. We endow  $\mathbf{P}_G$  with the topology arising from identification (via characteristic functions) of  $\mathbf{P}_G$  with  $\{0, 1\}^G$ . For  $K \in \mathbf{F}_G$  the sets

$$\{X \in \mathbf{P}_G : K \subseteq X\}, \quad \{X \in \mathbf{P}_G : X \cap K = \emptyset\}$$

form the sub-base of this topology.

After the topologization, each family  $\mathcal{F}$  of subsets of a group  $G$  can be considered as a subspace of  $\mathbf{P}_G$ , so one can ask about Borel complexity of  $\mathcal{F}$ , the question typical in the *Descriptive Set Theory* (see [1]). We ask these questions for the most intensively studied families in *Combinatorics of Groups*. For the origins of the families defined in section 1, see the survey [2]. The main results are in section 2 and its applications to  $\beta G$ , the Stone-Čech compactification of a discrete group  $G$ , in section 3. We conclude the paper with some comments.

## 1. DIVERSITY OF SUBSETS OF A GROUP

A subset  $A$  of a group  $G$  is called

- *large* if  $G = FA$  for some  $F \in \mathbf{F}_G$ ;
- *extralarge* if  $A \cap L$  is large for each large subset  $L$ ;
- *small* if  $L \setminus A$  is large for each large subset  $L$ ;
- *thick* if, for any  $F \in \mathbf{F}_G$  there exist  $g \in G$  such that  $Fg \subseteq A$ ;
- *prethick* if  $FA$  is thick for some  $F \in \mathbf{F}_G$ .

Some evident or easily verified (see [3]) relationships:  $A$  is large if and only if  $G \setminus A$  is not thick,  $A$  is small if and only if  $A$  is not prethick if and only if  $G \setminus A$  is extralarge. The family of all small subsets of  $G$  is an ideal in  $\mathbf{P}_G$ .

A subset  $A$  of a group  $G$  is called

- *P-small* if there exists an injective sequence  $(g_n)_{n \in \omega}$  in  $G$  such that the subsets  $\{g_n A : n \in \omega\}$  are pairwise disjoint;
- *weakly P-small* if, for any  $n \in \omega$ , there exists  $g_0, \dots, g_n$  such that the subsets  $g_0 A, \dots, g_n A$  are pairwise disjoint;
- *almost P-small* if there exists an injective sequence  $(g_n)_{n \in \omega}$  in  $G$  such that  $g_n A \cap g_m A$  is finite for all distinct  $n, m$ ;
- *near P-small* if, for every  $n \in \omega$ , there exists  $g_0, \dots, g_n$  such that  $g_i A \cap g_j A$  is finite for all distinct  $i, j \in \{0, \dots, n\}$ .

Every infinite group  $G$  contains a weakly  $P$ -small set, which is not  $P$ -small, see [4]. Each almost  $P$ -small subset can be partitioned into two  $P$ -small subsets [5]. Every countable Abelian group contains a near  $P$ -small subset which is neither weakly nor almost  $P$ -small [6].

A subset  $A$  of a group  $G$  with the identity  $e$  is called

- *thin* if  $gA \cap A$  is finite for each  $g \in G \setminus \{e\}$ ;
- *sparse* if, for every infinite subset  $Y$  of  $G$ , there exists a non-empty finite subset  $F \subset Y$  such that  $\bigcap_{g \in F} gA$  is finite.

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The union of two thin subsets need not to be thin, but the family of all sparse subsets is an ideal in  $\mathbf{P}_G$  [7]. For plenty of modifications and generalizations of thin and sparse subsets see [5], [8], [9], [10], [11].

## 2. RESULTS

For a group  $G$ , we denote by  $\mathbf{L}_G$ ,  $\mathbf{EL}_G$ ,  $\mathbf{S}_G$ ,  $\mathbf{T}_G$ ,  $\mathbf{PT}_G$  the sets of all large, extralarge, small, thick and prethick subsets of  $G$ , respectively.

**Theorem 1.** *For a countable group  $G$ , we have:  $\mathbf{L}_G$  is  $F_\sigma$ ,  $\mathbf{T}_G$  is  $G_\delta$ ,  $\mathbf{PT}_G$  is  $G_{\delta\sigma}$ ,  $\mathbf{S}_G$  and  $\mathbf{EL}_G$  are  $F_{\sigma\delta}$ .*

*Proof.* We take  $F, H \in \mathbf{F}_G$  and prove the following auxiliary claim: the set  $T(F, H) = \{A \in \mathbf{P}_G : H \subseteq FA\}$  is open.

Indeed, let  $F = \{g_1, \dots, g_n\}$  and  $(H_1, \dots, H_n)$  is a partition of  $H$ . Then the set  $\{A \in \mathbf{P}_G : H_1 \subseteq g_1 A, \dots, H_n \subseteq g_n A\}$  is open. It follows that  $T(F, H)$  is open.

Now, the set  $T_H(F) = \bigcup \{T(F, Hg) : g \in G\}$  is open and the set  $T(F) = \bigcap \{T_H(F) : H \in \mathbf{F}_G\}$  is  $G_\delta$ . We note that  $\mathbf{T}_G = T(\{e\})$  and  $\mathbf{PT}_G = \bigcup \{T(F) : F \in \mathbf{F}_G\}$  so  $\mathbf{T}_G$  is  $G_\delta$  and  $\mathbf{PT}_G$  is  $G_{\delta\sigma}$ .

Since  $\mathbf{S}_G = \mathbf{P}_G \setminus \mathbf{PT}_G$ ,  $\mathbf{S}_G$  is  $F_{\delta\sigma}$ . The mapping defined by  $A \mapsto G \setminus A$  is a homeomorphism of  $\mathbf{P}_G$ , so  $\mathbf{L}_G$  is homeomorphic to  $\mathbf{P}_G \setminus \mathbf{T}_G$  and  $\mathbf{EL}_G$  is homeomorphic to  $\mathbf{S}_G$ . Hence,  $\mathbf{L}_G$  is  $F_\sigma$  and  $\mathbf{EL}_G$  is  $F_{\delta\sigma}$ .  $\square$

**Theorem 2.** *For a countable group  $G$ , the sets of thin, weakly  $P$ -small and near  $P$ -small subsets of  $G$  are  $F_{\delta\sigma}$ .*

*Proof.* Given  $F \in \mathbf{F}_G$  and  $g \in G \setminus \{e\}$ , the set  $X(F, g) = \{A \in \mathbf{P}_G : ga \notin A \text{ for each } a \in A \setminus F\}$  is closed. The set  $X(g) = \bigcup \{X(F, g) : F \in \mathbf{F}_G\}$  is  $F_\sigma$ , and  $\bigcap \{X(g) : g \in G \setminus \{e\}\}$  is the set of all thin subsets.

For  $n \in \omega$ ,  $[G]^n$  denotes the family of all  $n$ -subsets of  $G$ . Given  $F \in [G]^n$ , the set  $Y(F) = \{A \in \mathbf{P}_G : gA \cap hA = \emptyset \text{ for all distinct } g, h \in F\}$  is closed, and the set of all weakly  $P$ -small subsets of  $G$  coincides with

$$\bigcap_{n \in \omega} \bigcup \{Y(F) : F \in [G]^n\}.$$

Given  $F \in [G]^n$  and  $H \in \mathbf{F}_G$ , the set  $Y(F, H) = \{A \in \mathbf{P}_G : g(A \setminus H) \cap h(A \setminus H) = \emptyset \text{ for all distinct } g, h \in F\}$  is closed, and the set of all near  $P$ -small subsets of  $G$  coincides with

$$\bigcap_{n \in \omega} \bigcup \{Y(F, H) : F \in [G]^n, H \in \mathbf{F}_G\}.$$

$\square$

We recall that a topological space  $X$  is *Polish* if  $X$  is homeomorphic to a separable complete metric space. A subset  $A$  of a Polish space  $X$  is *analytic* if  $A$  is a continuous image of some Polish space, and  $A$  is *coanalytic* if  $X \setminus A$  is analytic.

Using the classical tree technique [1] adopted to groups in [10], we get.

**Theorem 3.** *For a countable group  $G$ , the ideal of sparse subsets is coanalytic and the set of  $P$ -small subsets is analytic in  $\mathbf{P}_G$ .*

## 3. APPLICATIONS TO $\beta G$

Given a discrete group  $G$ , we identify the Stone-Ćech compactification  $\beta G$  with the set of all ultrafilters on  $G$  and consider  $\beta G$  as a right-topological semigroup (see [12]). Each non-empty closed subspace  $X$  of  $\beta G$  is determined by some filter  $\varphi_X$  on  $G$ :

$$X = \bigcap \{\overline{\Phi} : \Phi \in \varphi_X\}, \quad \overline{\Phi} = \{p \in \beta G : \Phi \in p\}.$$

On the other hand, each filter  $\Phi$  on  $G$  is a subspace of  $\mathbf{P}_G$ , so we can ask about complexity of  $X$  as the complexity of  $\varphi_X$  in  $\mathbf{P}_G$ .

The semigroup  $\beta G$  has the minimal ideal  $K_G$  which play one of the key parts in combinatorial applications of  $\beta G$ . By [3] Theorem 1.5, the closure  $cl(K_G)$  is determined by the filter of all extralarge subsets of  $G$ . If  $G$  is countable, applying Theorem 1, we conclude that  $cl(K_G)$  has the Borel complexity  $F_{\sigma\delta}$ .

An ultrafilter  $p$  on  $G$  is called *strongly prime* if  $p \notin cl(G^*G^*)$ , where  $G^*$  is a semigroup of all free ultrafilters on  $G$ . We put  $X = cl(G^*G^*)$  and choose the filter  $\varphi_X$  which determine  $X$ . By [7],  $A \in \varphi_X$  if and only if  $G \setminus A$  is sparse. If  $G$  is countable, applying Theorem 3, we conclude that  $\varphi_X$  is coanalytic in  $\mathbf{P}_G$ .

Let  $(g_n)_{n \in \omega}$  be an injective sequence in  $G$ . The set

$$\{g_{i_1}g_{i_2} \dots g_{i_n} : 0 \leq i_1 < i_2 < \dots < i_n < \omega\}$$

is called an *FP-set*. By the Hindman Theorem 5.8 [12], for every finite partition of  $G$ , at least one cell of the partition contains an *FP-set*. We denote by  $\mathbf{FP}_G$  the family of all subsets of  $G$  containing some *FP-set*. A subset  $A$  of  $G$  belongs to  $\mathbf{FP}_G$  if and only if  $A$  is an element of some idempotent of  $\beta G$ . By analogy with Theorem 3, we can prove that  $\mathbf{FP}_G$  is analytic in  $\mathbf{P}_G$ .

#### 4. COMMENTS AND OPEN QUESTIONS

1. Answering a question from [13], Zakrzewski proved [14] that, for a countable amenable group  $G$ , the ideal of absolute null subsets has the Borel complexity  $F_{\sigma\delta}$ . Each absolute null subset is small but, for every  $\epsilon > 0$ , there exists a small subset  $A$  of  $G$  such that  $\mu(A) > 1 - \epsilon$  for some Banach measure  $\mu$  on  $G$  (see [5]).

2. The classification of subsets of a group by their size can be considered in much more general context of *Asymptology* (see [15]). In this context, large, thick and small subsets play the parts of dense, open and nowhere dense subsets of a uniform topological space. For dynamical look at the subsets of a group see [16].

3. The following type of subsets of a group arised in Asymptology [11]. A subset  $A$  of a group  $G$  is called *scattered* if  $A$  has no subsets coarsely equivalent to the Cantor macrocube. Each sparse subset is scattered, each scattered subset is small and the set of all scattered subsets of  $G$  is an ideal in  $\mathbf{P}_G$ .

By Theorem 1 [11], a subset  $A$  of a group  $G$  is scattered if and only if  $A$  contains no piecewise shifted *FP*-sets.

Let  $(g_n)_{n \in \omega}$  be an injective sequence in  $G$  and let  $(b_n)_{n \in \omega}$  is a sequence in  $G$ .

The set

$$\{g_{i_1}g_{i_2} \dots g_{i_n}b_{i_n} : 0 \leq i_1 < i_2 < \dots < i_n < \omega\}$$

is called a *piecewise shifted FP-set*.

Using this combinatorial characterization and the tree technique from [10], we can prove that the ideal of scattered subsets of a countable group  $G$  is coanalytic in  $\mathbf{P}_G$ .

4. By [17], every meager topological group  $G$  can be represented as the product  $G = CN$  of some countable subset  $C$  and nowhere dense subset  $N$ . Every infinite group  $G$  can be represented as a union of some countable family of small subsets [3].

*Can every infinite group  $G$  be represented as the product  $G = CS$  of some countable subset  $C$  and small subset  $S$ ?*

The answer is positive if either  $G$  is amenable or  $G$  has a subgroup of countable index.

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